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ON A DIMENSIONAL REDUCTION METHOD.

SOME APPROXIMATION-THEORETIC RESULTS.

by

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and

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Abstract

This paper is the second in a series of three that analyze a method of dimensional reduction. It contains some results for approximation of functions on the interval [-1,1] with elements from the nullspace of P^N, $N \geq 1$, where P is a second order ordinary differential operator. A special case of this is approximation by polynomials.

The one-dimensional results are used as a tool to prove similar versions in several dimensions. These multi-dimensional results are directly related to the approximate method of dimensional reduction that was introduced in [13], and they lead to statements about the convergence properties of this approach.

The third paper, which analyzes the adaptive aspects of the method, is forthcoming.

1. Introduction

In a recent paper, [13], we introduced the concept of dimensionally reduced solutions to an elliptic boundary value problem. These are obtained by projecting (in the energy) the true solution of the boundary value problem in the n+1-dimensional domain $\omega \times [-h,h]$ onto spaces of the form

$$V_N^h = \{ \sum_{j=0}^{N} w_j(\underline{x})\phi_j(y/h) \,|\, w_j \text{ arbitrary}\} \quad ,$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a given set of functions on $[-1,1]$, ($\underline{x}$ are coordinates on ω and y ranges over $[-h,h]$). For some basic ideas behind this concept, see the introduction to [13]. In that paper the focus was on the right selection of the ϕ_j's. It was shown there that for a very wide class of problems the ϕ_j's should be selected such that

$$\text{span}\{\phi_j\}_{j=0}^{2k-1} = N(P^k) \quad ,$$

where P is a second order differential operator intrinsic to the elliptic boundary value problem.

The estimates of the error given in [13] were asymptotic in $h \to 0$. The present paper, which was already announced there, treats convergence as $N \to \infty$ for a fixed value of h. For convenience the fixed value of h is set equal to 1.

If the bilinear form associated with the elliptic boundary value problem satisfies some kind of "inf-sup" condition, then it is well known that the rate of convergence is the same as the rate of approximation (cf.[1]).

The results proven here are hence formulated as approximation-theoretic estimates, and as such have interest regardless of the concept of dimensionally reduced solutions.

The results are all concerning approximation in the L^2- and H^1-norms, i.e. ideally suited for second order problems. This is not crucial and similar results can also be obtained, e.g., for the norms $(\int_{-1}^{1}||\frac{du}{dy}||^2dy + \int_{-1}^{1}||A^{1/2}u||^2dy)^{1/2}$ introduced in [13]. (A here denotes a strictly positive definite (unbounded) operator in a Hilbert space $\mathcal{H}$, and u is a function with values in $\mathcal{H}$.)

For reasons of convenience the approximation results are formulated without any boundary conditions. Various types of fixed boundary conditions can immediately be included based on the present proofs.

Estimates of the error introduced by dimensional reduction, as N goes to ∞, do exist in the literature (cf. [5,7]). The problems considered in those two papers come from structural mechanics. The elliptic operators have constant coefficients, i.e. the ϕ_j's are polynomials. The results are not nearly as strong as the ones established here. In [7] the estimates are based on the degree of regularity in C^k-spaces; this is not very well suited to the regularity properties of solutions to elliptic boundary value problems, and therefore gives crude estimates. The estimates in [5] are based on bounding the remainder in the N-th order Taylor expansion. The estimates are very crude and do not give any indication of the rate of approximation.

We now give a short review of the contents of this paper. In section 2 it is shown that the set $\bigcup_{k=1}^{\infty} \mathcal{N}(P^k)$ ($\mathcal{N}$ denotes the nullspace) is dense in H^1 for any second order operator $P = b\frac{d}{dy}a\frac{d}{dy}$, where both a and b are bounded from above and away from 0. This is the obvious generalization of the fact that the polynomials are dense in H^1, and it also justifies the claim that the dimensionally reduced solutions introduced in [13] will get arbitrarily close to the true solution. In section 3 the rate of approximation using functions in $\mathcal{N}(P^N)$, $N \geq 1$, is linked to the regularity of u in spaces of the type $\mathcal{D}(P^m)$. This general result though is not always optimal, as shown, e.g., by Theorem 4.1. Section 4

is devoted to giving a necessary and sufficient condition for a certain rate of approximation by polynomials (i.e., the case where the operator P is a constant-coefficient operator). In section 5 this is carried over to results in several dimensions -- directly relating to the concept of dimensional reduction. The example treated in section 6 is of the same type as the numerical examples in [13]. Finally the appendix contains the proofs of several results about the eigenvalues and eigenfunctions for two-point boundary value problems, as used in sections 2 and 3.

Note: Unless otherwise stated, all constants denoted by capital letters are generic.

2. A Density Result

Let a and b be two functions in $L^{\infty}([-1,1])$ such that $\exists$ constants a_o, b_o with

$$0 < a_o \leq a(y)$$
$$0 < b_o \leq b(y) \quad .$$

By P we denote the differential operator

$$b \frac{d}{dy} a \frac{d}{dy} \quad .$$

P is considered as a mapping $L^2([-1,1]) \supseteq \mathcal{D}(P) \to L^2([-1,1])$. $\mathcal{N}(P^k)$ denotes the null-space of the operator P^k for any integer $k \geq 1$. It is easily seen that $\mathcal{N}(P^k) \subseteq H^1([-1,1])$. The first theorem in this section proves the density of a certain class of functions associated with the operator P .

Theorem 2.1

$\bigcup_{k=1}^{\infty} \mathcal{N}(P^k)$ is dense in $H^1([-1,1])$

Proof

By a change of variables, $y' = \int_{-1}^{y} \frac{1}{b(s)} ds$, and multiplication by -1 the operator P transforms into

$$- \frac{d}{dy'} \frac{a}{b} \frac{d}{dy'} \quad .$$

We can therefore for the proof of this theorem assume that P is given by $- \frac{d}{dy} a(y) \frac{d}{dy}$ where a satisfies: $\exists$ a constant a_o with $0 < a_o \leq a(y)$.

Define the operator Q by

$\mathcal{D}(Q) = \mathcal{D}(P) \cap \overset{o}{H}{}^1([-1,1])$ and $Q = P$ on $\mathcal{D}(Q)$. Let f_0 denote the function

$$f_0(y) = \int_{-1}^{y} \frac{1}{a(s)}\, ds \in N(P) \quad ,$$

and define the sequence $\{f_i\}_{i=0}^{\infty}$ by

$$f_i = Q^{-i} f_0 \in N(P^{i+1}) \quad .$$

$0 < \lambda_o \leq \lambda_1 \leq \;.\;.\;.\leq \lambda_m \leq \lambda_{m+1}\;.\;.$ denotes the eigenvalues of Q (repeated according to multiplicity). Let $\{u_m\}_{m=0}^{\infty}$ be an *orthonormal* basis of eigenfunctions, u_m corresponding to λ_m. f_0 can then be expanded as

$$f_0 = \sum_{m=0}^{\infty} \alpha_m u_m \quad ,$$

and with this notation

$$f_i = \sum_{m=0}^{\infty} \alpha_m \lambda_m^{-i} u_m \quad .$$

We now proceed to prove that any eigenfunction u_m can be approximated from within $\bigcup_{k=1}^{\infty} N(P^k)$. The proof is by induction in m , and we start with $m = 0$. For any $i \geq 1$ we have that

$$\begin{aligned}
&||u_0 - \alpha_0^{-1} \lambda_0^{i} f_i||_{H^1}^2 \leq \\
&C||Q^{1/2}(u_0 - \alpha_0^{-1} \lambda_0^{i} f_i)||_{L^2}^2 = \\
&C\, \lambda_0 \sum_{j=1}^{\infty} \left(\alpha_j/\alpha_0\right)^2 \left(\lambda_0/\lambda_j\right)^{2i-1} \leq C \left(\lambda_0/\alpha_0^2\right) \left(\lambda_0/\lambda_1\right)^{2i-1} \sum_{j=1}^{\infty} \alpha_j^2 \quad ,
\end{aligned}$$

where we have used Lemma A.3 to guarantee that $\alpha_0 \neq 0$. Since $\sum_{j=1}^{\infty} \alpha_j^2 < \infty$ and, by Lemma A.1, $\lambda_0/\lambda_1 < 1$ this shows that

$$\alpha_0^{-1} \lambda_0^i f_i \to u_0 \qquad \text{as } i \to \infty \text{, in } H^1([-1,1]) \;,$$

or
$$u_0 \in \overline{\bigcup_{k=1}^{\infty} N(P^k)}$$

(———— denotes the closure in H^1) .

Now assume it has been proven for some $m \geq 1$ that

$$\{u_j\}_{j=0}^{m-1} \subseteq \overline{\bigcup_{k=1}^{\infty} N(P^k)} \quad .$$

From Lemma A.3 we know that $\alpha_m \neq 0$, and hence for any $i \geq 1$

$$u_m - \alpha_m^{-1} \lambda_m^i f_i = x_{m,i} - \sum_{j=m+1}^{\infty} \left(\alpha_j/\alpha_m\right)\left(\lambda_m/\lambda_j\right)^i u_j \quad ,$$

where $x_{m,i} = -\sum_{j=0}^{m-1} \left(\alpha_j/\alpha_m\right)\left(\lambda_m/\lambda_j\right)^i u_j \in \overline{\bigcup_{k=1}^{\infty} N(P^k)}$

due to the induction hypothesis. As before the H^1 - norm of the sum

$$\sum_{j=m+1}^{\infty} \left(\alpha_j/\alpha_m\right)\left(\lambda_m/\lambda_j\right)^i u_j$$

can be estimated by

$$C\left(\sqrt{\lambda_m}/\alpha_m\right)\left(\lambda_m/\lambda_{m+1}\right)^{i-1/2} \left(\sum_{j=m+1}^{\infty} \alpha_j^2\right)^{1/2} \quad .$$

Because of the facts that $\left(\sum_{j=m+1}^{\infty} \alpha_j^2\right)^{1/2} < \infty$ and, by Lemma A.1, $\lambda_m/\lambda_{m+1} < 1$ this shows that

$$x_{m,i} + \alpha_m^{-1}\lambda_m^i f_i \to u_m \quad \text{as} \quad i \to \infty ,$$

in $H^1([-1,1])$, i.e.

$$\{u_j\}_{j=0}^m \subseteq \overline{\bigcup_{k=1}^{\infty} N(P^k)} .$$

This finishes the induction proof, and we conclude that

$$\{u_j\}_{j=0}^\infty \subseteq \overline{\bigcup_{k=1}^{\infty} N(P^k)} .$$

From the definition of Q it immediately follows that

$$\mathcal{D}(Q^{1/2}) = \overset{o}{H}{}^1([-1,1]) ,$$

and since $\{u_j\}_{j=0}^\infty$ is complete in $\mathcal{D}(Q^{1/2})$ this proves that

$$\overset{o}{H}{}^1([-1,1]) \subseteq \overline{\bigcup_{k=1}^{\infty} N(P^k)}$$

Now if $u \in H^1([-1,1])$ we shall, by choosing $c = u(-1)$ and $d = (u(1)-u(-1))/\int_{-1}^{1} a^{-1}(s)ds$, obtain that

$$u - c - df_0 \in \overset{o}{H}{}^1([-1,1]) .$$

Since $1, f_0 \in N(P)$ we see by a combination of this and the previously proven inclusion that

$$H^1([-1,1]) = \overline{\bigcup_{k=1}^{\infty} N(P^k)}$$

$\square$

Based on Theorem 2.1 we can easily prove a result concerning the dimensionally reduced solutions as introduced in [13]. This result guarantees the fulfilment of the goal stating that the dimensionally reduced solutions shall be able to get arbitrarily close to the true solution.

Let ω denote a domain in $\mathbb{R}^n$ with a Lipschitz boundary.

<u>Theorem 2.2</u>

The set

$$\{ \sum_{j=0}^{J} v_j(\underline{x})q_j(y) \,|\, J \in \mathbb{N}, v_j \in H^1(\omega) \text{ and } q_j \in \bigcup_{k=1}^{\infty} N(P^k) \text{ for } 0 \le j \le J \}$$

is dense in $H^1(\omega \times [-1,1])$.

<u>Proof</u>

Follows immediately from Theorem 2.1 and the fact that

$$\{ \sum_{j=0}^{J} v_j(\underline{x})w_j(y) \,|\, J \in \mathbb{N}, v_j \in H^1(\omega) \text{ and } w_j \in H^1([-1,1]) \text{ for } 0 \le j \le J \}$$

is dense in $H^1(\omega \times [-1,1])$.

$\square$

3. Estimates of the Rate of Approximation

In the previous section we proved the density of a certain class of functions associated with the operator $P = b\frac{d}{dy} a \frac{d}{dy}$. In this section we shall prove some results concerning the rate of approximation. The first theorem is the following.

Theorem 3.1

Assume that $a,b \in C^2([-1,1])$, and let m be an integer ≥ 0. For any $\varepsilon > 0$ there exists a constant C_ε such that

$$\inf_{v \in N(P^N)} ||u-v||_{L^2} \leq C_\varepsilon \, N^{-m+\varepsilon} \, ||u||_{\mathcal{D}(P^m)} \qquad \forall N \geq 1 \quad .$$

Note. $||\cdot||_{\mathcal{D}(P^m)}$ denotes the norm $||P^m(\cdot)||_{L^2} + ||\cdot||_{L^2}$.

One can of course combine the statement in Theorem 3.1 with interpolation by the K-method (cf. [4]). This way it follows, that if $u \in (L^2;\mathcal{D}(P^m))_{s,\infty}$ for some $0 < s < 1$, then for any $\varepsilon > 0$

$$\inf_{v \in N(P^N)} ||u-v||_{L^2} \leq C_\varepsilon \, N^{-ms+\varepsilon} \, ||u||_{(L^2,\mathcal{D}(P^m))_{s,\infty}} \quad .$$

The smoothness requirement that $a,b \in C^2([-1,1])$ is not necessary; as it immediately will follow from this proof we only need that a/b is a C^2-function. This last remark applies to all of the results in this section.

In order to prove Theorem 3.1 we need an auxiliary result concerning uniform approximation by polynomials. This result can be found, e.g., in chapter 6 of [6].

Lemma 3.1

Let ϕ be a function in $C^0([c,d])$. Define $\hat{\phi}$ by $\hat{\phi}(t) = \phi(\frac{c-d}{2}\cos t + \frac{c+d}{2})$, $t \in [0,\pi]$.

Let r be a non-negative integer. There exists a constant C_r such that for any ϕ with $\hat{\phi} \in C^r([0,\pi])$ the following estimate holds

$$\inf_{p_N} |\phi - p_N|_0 \le C_r (N+1)^{-r} |\hat{\phi}|_r \quad \forall N \ge 0 \quad .$$

The infinum here is taken over all polynomials p_N of degree $\le N$. $|\cdot|_0$ and $|\cdot|_r$ denote the norms in $C^0([c,d])$ and $C^r([o,\pi])$ respectively.

We now continue with

Proof of Theorem 3.1

Like in the proof of Theorem 2.1 we may also here assume that P is given by $P = -\frac{d}{dy}\, a(y)\, \frac{d}{dy}$. Let f_i $i \ge 0$ be defined as in that same proof. For any set of coefficients $\{c_i\}_{i=0}^N$ we have, using Lemma A.3, that

$$u - \sum_{i=0}^N c_i f_i = \sum_{j=0}^\infty \alpha_j (\beta_j/\alpha_j - \sum_{i=0}^N c_i \lambda_j^{-i}) u_j \quad ,$$

where $\sum_{j=0}^\infty \beta_j u_j$ is the expansion corresponding to u. If by p_N we denote the polynomial $p_N(x) = \sum_{i=0}^N c_i x^i$, then the above can be rewritten as

$$u - \sum_{i=0}^N c_i f_i = \sum_{j=0}^\infty \alpha_j (\beta_j/\alpha_j - p_N(\lambda_j^{-1})) u_j \quad ,$$

and this leads to the equality

$$(1) \qquad ||u - \sum_{i=0}^N c_i f_i||_{L^2}^2 = \sum_{j=0}^\infty \alpha_j^2 (\beta_j/\alpha_j - p_N(\lambda_j^{-1}))^2 \quad .$$

As in the proof of Theorem 2.1 Q denotes the restriction of P to $\mathcal{D}(P) \cap \overset{o}{H}{}^1([-1,1])$. Let us now for a while assume that $u \in \mathcal{D}(Q^k)$.

Choose Λ so that $\{\lambda_j^{-1}\}_{j=0}^{\infty} \subseteq [0,\Lambda]$. Define a sequence of functions $\phi_M \in C^{\infty}([0,\Lambda])$, $1 \leq M$, with the following properties

$$\phi_M(\lambda_j^{-1}) = \beta_j/\alpha_j \qquad 0 \leq j \leq M-1$$

$$\phi_M(x) \quad = 0 \qquad \text{on} \quad [0,\lambda_M^{-1}] \quad .$$

Let Φ denote the mapping

$$\Phi(t) = \frac{\Lambda}{2}\,(1-\cos t):\quad [0,\pi] \to [0,\Lambda] \ ,$$

it then follows from Lemma A.1 and Lemma A.2 that

$$|\Phi^{-1}(\lambda_{j-1}^{-1}) - \Phi^{-1}(\lambda_j^{-1})| \geq C/j^2 \text{ for any } j \geq 1 \ .$$

This estimate tells us that it is possible to construct the ϕ_M's such that $\forall M,\ j \geq 1$

$$|\phi_M(\Phi(t))|_r \leq C_r \sup_{0 \leq j \leq M-1} |(j+1)^{2r}\beta_j/\alpha_j| \quad .$$

Now since $u \in \mathcal{D}(Q^k)$ we know that $|\beta_j| \leq C_k(j+1)^{-2k}||Q^k u||_{L^2}$, and combining this with Lemma A.4 we get

$$|\beta_j/\alpha_j| \leq C_k(j+1)^{-2k+1}||Q^k u||_{L^2}$$

i.e., we have for any $r \geq k$ and $M \geq 1$

$$| \phi_M(\Phi(t)) |_r \leq C_r M^{2(r-k)+1} ||Q^k u||_{L^2} \quad .$$

Because of Lemma 3.1 we can now, for any $r \geq k$, $M \geq 1$, find a polynomial p_N^M of degree $\leq N$ such that

$$|\phi_M - p_N^M|_0 \leq C_r (N+1)^{-r} M^{2(r-k)+1} ||Q^k u||_{L^2}$$

We now go back to estimate the right hand side of (1)

$$\sum_{j=0}^{\infty} \alpha_j^2 (\beta_j/\alpha_j - p_N^M(\lambda_j^{-1}))^2 \leq \sum_{j=0}^{M-1} \alpha_j^2 ((\phi_M - p_N^M)(\lambda_j^{-1}))^2 + 2 \sum_{j=M}^{\infty} \beta_j^2 +$$

$$+ 2 \sum_{j=M}^{\infty} \alpha_j^2 ((\phi_M - p_N^M)(\lambda_j^{-1}))^2$$

The first and the third sum can be estimated by

$$C_r (N+1)^{-2r} M^{4(r-k)+2} \left(\sum_{j=0}^{\infty} \alpha_j^2 \right) ||Q^k u||_{L^2}^2 \leq$$

$$\leq C_r (N+1)^{-2r} M^{4(r-k)+2} ||Q^k u||_{L^2}^2 \quad ,$$

the second by

$$C_k \sum_{j=M}^{\infty} (j+1)^{-4k} ||Q^k u||_{L^2}^2 \leq C_k M^{-4k+1} ||Q^k u||_{L^2}^2 \quad .$$

In summary we have therefore proven

$$||u - \sum_{i=0}^{N} c_i f_i||^2_{L^2} \leq C_r((N+1)^{-2r} M^{4(r-k)+2} + M^{-4k+1})||Q^k u||^2_{L^2}$$

for any $r \geq k$, $N \geq 0$ and $M \geq 1$. Taking $M = [\sqrt{N}] + 1$ ($[\cdot]$ denotes the integer part) this estimate gives

$$(2) \qquad ||u - \sum_{i=0}^{N} c_i f_i||^2_{L^2} \leq C_k(N+1)^{-2k+1}||Q^k u||^2_{L^2} ,$$

all provided that $u \in \mathcal{D}(Q^k)$. Let Π_N denote the L^2-projection onto linear combinations of the functions $f_0 \ldots, f_N$.

(2) expresses that

$$||u-\Pi_N u||_{L^2} \leq C_k(N+1)^{-k+1/2}||Q^k u||_{L^2} ,$$

at the same time it is clear that

$$||u-\Pi_N u||_{L^2} \leq ||u||_{L^2} .$$

Applying interpolation by the K-method we get for any $0 \leq m \leq k$

$$||u-\Pi_N u||_{L^2} \leq C_k(N+1)^{-(k-1/2)\frac{m}{k}}||Q^m u||_{L^2} .$$

Now let m be fixed and $k \to \infty$, from the previous inequality we then get

$$\forall \varepsilon > 0 \;\; \exists C_\varepsilon \quad \text{such that}$$

(3) $$||u-\Pi_N u||_{L^2} \leq C_\varepsilon (N+1)^{-m+\varepsilon} ||Q^m u||_{L^2} \quad ,$$

provided $u \in \mathcal{D}(Q^m)$. If we only know that $u \in \mathcal{D}(P^m)$, then choose $\{g_j\}_{j=1}^m \subseteq N(P^m)$ such that

$$P^{j-1} g_j = P^{j-1} u \qquad \text{for} \qquad y = \pm 1$$

$$P^i g_j = 0 \qquad \text{for} \qquad y = \pm 1$$

and any $i \neq j-1$, (this is obviously possible). This way

$$u - \sum_{j=1}^m g_j \in \mathcal{D}(Q^m) \qquad \text{and}$$

(4) $$\sum_{j=1}^m g_j \in N(P^m) \quad .$$

From (3) and (4) it now follows that

$$||u - \Pi_N(u - \sum_{j=1}^m g_j) - \sum_{j=1}^m g_j||_{L^2} \leq C_\varepsilon (N+1)^{-m+\varepsilon} \cdot$$

$$\cdot ||Q^m(u - \sum_{j=1}^m g_j)||_{L^2} = C_\varepsilon (N+1)^{-m+\varepsilon} ||P^m u||_{L^2} \quad .$$

Since the image under Π_N is contained in $N(P^{N+1})$, this estimate yields the desired result for $N \geq m$. There are only a finite number of N's $< m$, and hence the result can be obtained for all N by possibly increasing C_ε .

$\square$

Based on Theorem 3.1 we can prove the following result concerning approximation in the H^1-norm.

<u>Theorem 3.2</u>

Assume that $a,b \in C^2([-1,1])$, and let m be an integer ≥ 1. For any $\varepsilon > 0$ there exists a constant C_ε such that

$$\inf_{v \in N(P^N)} ||u-v||_{H^1} \leq C_\varepsilon \; N^{-m+1+\varepsilon} \; ||u||_{\mathcal{D}(P^m)} \qquad \forall N \geq 1$$

<u>Proof</u>

From Theorem 3.1 it follows that there exist $\tilde{v}_N \in N(P^N)$ such that

$$||Pu-\tilde{v}_N||_{L^2} \leq C_\varepsilon \; N^{-m+1+\varepsilon} \; ||u||_{\mathcal{D}(P^m)} \quad .$$

Now choose $v_N \in N(P^{N+1})$ with $Pv_N = \tilde{v}_N$, and such that $v_N = u$ for $y = \pm 1$. It then follows that

$$||u-v_N||_{H^1} \leq C||Pu-Pv_N||_{L^2} = C||Pu-\tilde{v}_N||_{L^2} \leq C_\varepsilon \; N^{-m+1+\varepsilon} \; ||u||_{\mathcal{D}(P^m)}$$

$\square$

We can also easily prove a result relating to the dimensionally reduced solutions as introduced in [13]. Let $\omega \subseteq \mathbb{R}^n$ be a domain with a Lipschitz boundary. $\underline{x} = (x_1,\ldots,x_n)$ denotes coordinates in ω and y ranges over $[-1,1]$. $\tilde{P}$ denotes $b(y) \frac{\partial}{\partial y} a(y) \frac{\partial}{\partial y}$ considered as an operator $L^2(\omega \times [-1,1]) \supseteq \mathcal{D}(\tilde{P}) \to L^2(\omega \times [-1,1])$.

<u>Theorem 3.3</u>

Assume that $a,b \in C^2([-1,1])$ and let m be an integer ≥ 1. Let u be an element of $L^2(\omega \times [-1,1])$ with $\frac{\partial}{\partial x_1}u, \ldots, \frac{\partial}{\partial x_n}u$, $u \in \mathcal{D}(\tilde{P}^m)$. Then for any $\varepsilon > 0$ there exist C_ε (independent of u) such that

$$\inf_{v \in V_N} ||u-v||_{H^1(\omega \times [-1,1])} \leq C_\varepsilon (N+1)^{-m+1+\varepsilon} \left(\sum_{i=1}^{n} ||\frac{\partial}{\partial x_i} u||_{\mathcal{D}(\tilde{P}^m)} + ||u||_{\mathcal{D}(\tilde{P}^m)} \right)$$

Here V_N denotes the set $\{\sum_{j=0}^{N} w_j(\underline{x})\phi_j(y) \,|\, w_j \in H^1(\omega)\}$, where $\{\phi_j\}_{j=0}^{2k-1}$ is a basis for $N(P^k) \subseteq H^1([-1,1])$.

<u>Proof</u>

Let v_N denote the orthogonal projection of $\tilde{P}u$ onto $\{\sum_{j=0}^{N} w_j(\underline{x})\phi_j(y) \,|\, w_j \in L^2(\omega)\}$ in the $L^2(\omega \times [-1,1])$ inner product. Then it is clear that $\frac{\partial v_N}{\partial x_i}$ is the L^2 projection of $\tilde{P}\frac{\partial}{\partial x_i}u$ onto the same subspace. From Theorem 3.1 we immediately get

$$\sum_{i=1}^{n} ||\tilde{P}\frac{\partial}{\partial x_i}(u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||\tilde{P}(u-v_N^*)||_{L^2(\omega \times [-1,1])} \leq$$

$$\leq C_\varepsilon (N+1)^{-m+1+\varepsilon} \left(\sum_{i=1}^{n} ||\frac{\partial}{\partial x_i} u||_{\mathcal{D}(\tilde{P}^m)} + ||u||_{\mathcal{D}(\tilde{P}^m)} \right) .$$

for any function $v_N^* \in V_{N+3}$ with $\tilde{P}v_N^* = v_N$ (if N is odd such a v_N^* will be contained in V_{N+2}, but this is not necessarily so for N even). Now choosing v_N^* so that also $v_N^* = u$ for $y = \pm 1$ (this is obviously possible), it follows that

$$\sum_{i=1}^{n} \left|\left|\frac{\partial}{\partial x_i} (u-v_N^*)\right|\right|_{L^2(\omega \times [-1,1])} + \left|\left|\frac{\partial}{\partial y} (u-v_N^*)\right|\right|_{L^2(\omega \times [-1,1])} +$$

$$+ \left|\left|u-v_N^*\right|\right|_{L^2(\omega \times [-1,1])} \leq$$

$$C\left(\sum_{i=1}^{n} \left|\left|\tilde{P} \frac{\partial}{\partial x_i} (u-v_N^*)\right|\right|_{L^2(\omega \times [-1,1])} + \left|\left|\tilde{P}(u-v_N^*)\right|\right|_{L^2(\omega \times [-1,1])}\right) \leq$$

$$C_\varepsilon (N+1)^{-m+1+\varepsilon} \left(\sum_{i=1}^{n} \left|\left|\frac{\partial}{\partial x_i} u\right|\right|_{\mathcal{D}(\tilde{P}^m)} + \left|\left|u\right|\right|_{\mathcal{D}(\tilde{P}^m)}\right) .$$

$\square$

4. The Constant Coefficient Case - 1 Dimensional Results

The following two sections are devoted to the case where the operator P has constant coefficients. In the previous section we estimated the rate of approximation for general P's , but the estimates established there do not have exact inverse counterparts nor are they always optimal. As will be shown in this section and the next, the question of approximation rate can be much further clarified when P is a constant coefficient operator. We start with an analysis of the 1 dimensional problem. The space $N(P^k)$, $k \geq 1$, consists simply of all polynomials of degree $\leq 2k-1$. Theorem 3.1 combined with interpolation says that if $u \in H^t([-1,1])$, then there exist polynomials p_N , of degree N , such that $||u-p_N||_{L^2} \leq C_\varepsilon (N+1)^{-t/2+\varepsilon}$. Under the present simplified circumstances we can prove a better result. In the formulation of this result we use the Besov spaces $B^t_{2,\infty}$, $t > 0$, (cf. [4]) , instead of the ordinary Sobolev spaces H^t . For an interpretation in terms of the spaces H^t use the inclusions $H^t \subseteq B^t_{2,\infty} \subseteq H^{t-\varepsilon}$ valid for any $t > 0$, $\varepsilon > 0$.

Theorem 4.1

Let t be a given positive number. There exists a constant C_t such that for any $u \in B^t_{2,\infty}([-1,1])$ one can find a sequence of polynomials $\{p_N\}^\infty_{N=0}$, the degree of $p_N \leq N$, with

$$||u-p_N||_{L^2} \leq C_t (N+1)^{-t} ||u||_{B^t_{2,\infty}} .$$

Note. A similar theorem is also valid for the H^1-norm. The estimate here becomes (for $t > 1$)

$$||u-p_N||_{H^1} \leq C_t (N+1)^{-t+1} ||u||_{B^t_{2,\infty}} .$$

The rate of approximation established in Theorem 4.1 is optimal in the following sense.

<u>Theorem 4.2</u>

If $u \in L^2([-1,1])$ and there exist a constant C and a sequence of polynomials $\{p_N\}_{N=0}^{\infty}$, the degree of $p_N \leq N$, such that

$$||u-p_N||_{L^2} \leq C(N+1)^{-t} \quad \text{for some} \quad t > 0 ,$$

then $u \in (B^t_{2,\infty})_{loc} \cap B^{t/2}_{2,\infty}$.

<u>Note</u>. Theorem 4.2 is not an exact inverse of Theorem 4.1 since it only guarantees that $u \in B^{t/2}_{2,\infty}([-1,1])$. But based on Theorem 4.2 we conclude that for a general type function in $B^t_{2,\infty}([-1,1])$ we cannot expect more than an approximation rate of $(N+1)^{-t}$.

Theorem 4.2 is optimal in that one can find u such that $||u-p_N|| \leq C_\varepsilon (N+1)^{-t+\varepsilon}$ and $u \notin B^{t/2+\varepsilon}_{2,\infty}$, $u \notin (B^{t+\varepsilon}_{2,\infty})_{loc}$ for any $\varepsilon > 0$ (cf. [12]) .

The proof of Theorem 4.1 is very simple, based on transforming u into a periodic function and estimating the remainder of the k'th order Fourier expansion. Details can be found, e.g., in [2].

The proof of Theorem 4.2 is not quite as simple. The cornerstone is the so-called Bernsteins inequality

$$||(\tfrac{d}{dy})p_N||_{L^2} \leq CN^2||p_N||_{L^2} \quad ,$$

valid for any polynomial of degree $\leq N$. For more details see [2] or [10].

As already noted Theorem 4.2, although optimal, is not an exact inverse of Theorem 4.1. This can be taken as evidence that the standard Sobolev or Besov spaces are not very good for expressing the kind of regularity needed for a

certain rate of approximation by polynomials. They do not take into account the well known fact, already noted by Timan (cf. [9]), that the polynomials have a certain ability to absorb singularities at the end points of an interval.

Let L denote the operator $-\frac{d}{dy}(1-y^2)\frac{d}{dy}$ with domain of definition

$$\mathcal{D}(L) = \{u \in L^2([-1,1]) \mid Lu \in L^2([-1,1])\} \quad .$$

Now introduce the Besov spaces H^t, $t > 0$, by

$$H^t = (\mathcal{D}(L^p);\mathcal{D}(L^q))_{s,\infty}$$

where p,q are two integers with $0 \leq p < t < q$ and $0 < s < 1$ is selected so that $p(1-s) + qs = t$. Because of Theorem 14.1 in [8], which says that $(\mathcal{D}(L^k);\mathcal{D}(L^\ell))_{\theta,2} = \mathcal{D}(L^{k(1-\theta)+\ell\theta})$, and the reiteration theorem on p.50 of [4] it follows that modulo equivalent norms H^t is independent of the choice of p and q .

We are now in a position to characterize completely the regularity needed for a certain order of approximation by polynomials.

<u>Theorem 4.3</u>

Let t be a positive number. For any $u \in H^t$ we can find a sequence of polynomials $\{p_N\}_{N=0}^{\infty}$, the degree of $p_N \leq N$, such that

$$||u-p_N||_{L^2} \leq (N+1)^{-2t}||u||_{H^t} \quad .$$

On the other hand if $u \in L^2([-1,1])$ and there exists a constant C_u and a sequence of polynomials $\{p_N\}_{N=0}^{\infty}$, the degree of $p_N \leq N$, such that

$$||u-p_N||_{L^2} \le C_u (N+1)^{-2t}$$

then $u \in \mathcal{H}^t$ and

$$||u||_{\mathcal{H}^t} \le C(C_u + ||u||_{L^2})$$

for some constant C independent of u .

<u>Note</u>. C_u here is not generic, it is the same constant in the two inequalities.

<u>Proof</u>

We start by proving the direct part. It is well known that the eigenfunctions of L are the Legendre polynomials $\{\ell_k\}_{k=0}^{\infty}$. Also

$$L(\ell_k) = k(k+1)\ell_k \quad .$$

Let now u be an element of $\mathcal{D}(L^p)$ and let $\sum_{m=0}^{\infty} \alpha_m \ell_m$ be the Legendre series for u . Since $u \in \mathcal{D}(L^p)$ we know that $\sum_{m=0}^{\infty} \alpha_m^2 m^{4p} < ||u||^2_{\mathcal{D}(L^p)}$. Define $p_N = \sum_{m=0}^{N} \alpha_m \ell_m$, then

$$||u-p_N||^2_{L^2} = \sum_{m=N+1}^{\infty} \alpha_m^2 \le (N+1)^{-4p} \sum_{m=N+1}^{\infty} \alpha_m^2 m^{4p} \le (N+1)^{-4p} ||u||^2_{\mathcal{D}(L^p)} ,$$

$$\text{i.e.,} \qquad ||u-p_N||_{L^2} \le (N+1)^{-2p} ||u||_{\mathcal{D}(L^p)} \quad .$$

Interpolation applied to this gives the desired result.

We now turn to the inverse. Assume that there exists polynomials p_N of degree $\le N$ such that

$$||u-p_N||_{L^2} \le C_u (N+1)^{-2t} \quad .$$

Define $\pi_o = p_1$, $\pi_m = p_{2^m} - p_{2^{m-1}}$ $m \ge 1$.

Then

$$||\pi_o||_{L^2} \le C_u + ||u||_{L^2} \quad \text{and}$$

$$||\pi_m||_{L^2} \le ||u-p_{2^m}||_{L^2} + ||u-p_{2^{m-1}}||_{L^2} \le C_t \cdot C_u 2^{-2tm} \quad m \ge 1 \quad .$$

Since $||\mathcal{L}p_n||_{L^2} \le Cn^2 ||p_n||_{L^2}$ for any polynomial p_n of degree $\le n$, it follows from above that for any non-negative integer q

$$||\mathcal{L}^q \pi_o||_{L^2} \le C_q (C_u + ||u||_{L^2}) \quad \text{and}$$

$$||\mathcal{L}^q \pi_m||_{L^2} \le C_q \cdot C_t \cdot C_u \cdot 2^{2(q-t)m} \quad m \ge 1 \quad .$$

Now define $v_k = \sum_{m=0}^{k} \pi_m$. We then get

$$||v_k||_{\mathcal{D}(\mathcal{L}^q)} \le \sum_{m=0}^{k} (||\mathcal{L}^q \pi_m||_{L^2} + ||\pi_m||_{L^2})$$

$$\le C_{q,t}(C_u + ||u||_{L^2} + C_u \sum_{m=1}^{k} 2^{2(q-t)m})$$

$$\le C_{q,t}\, 2^{2(q-t)k}(C_u + ||u||_{L^2})$$

provided $q > t$. At the same time

$$||u-v_k||_{L^2} = ||u-p_{2^k}||_{L^2} \leq C_u 2^{-2tk} \quad .$$

By defining $s_k = 2^{-2kq}$ we therefore have

$$s_k^{-\frac{t}{q}} \left(||u-v_k||_{L^2} + s_k ||v_k||_{\mathcal{D}(L^q)} \right) \leq C_{q,t} \left(C_u + ||u||_{L^2} \right) \quad ,$$

and since $s_k \to 0$ for $k \to \infty$ this proves that

$$u \in (L^2; \mathcal{D}(L^q))_{\frac{t}{q},\infty} = H^t \quad ,$$

with

$$||u||_{H^t} \leq C_{q,t}(C_u + ||u||_{L^2}) \quad .$$

$\square$

This theorem also allows a version formulated by using the spaces $\mathcal{D}(L^t)$ instead of the corresponding Besov spaces. It is derived from the inclusions $\mathcal{D}(L^t) \subseteq H^t \subseteq \mathcal{D}(L^{t-\varepsilon})$ valid for any $t > 0$, $\varepsilon > 0$.

A theorem similar to Theorem 4.3 but concerning approximation in the H^1 norm can be derived immediately based on Theorem 4.3.

For practical purposes, in determining the rate of approximation, the following characterization of $\mathcal{D}(L^q)$ (cf. [3]) will often be convenient:

$$\mathcal{D}(L^q) = \{u \in L^2([-1,1]) \,|\, u \in H^q([-1,1]) \ , \ (1-y^2)^q u \in H^{2q}([-1,1])\} \quad .$$

Let us now give one simple example that shows how a result similar to Theorem 4.3 can be established also in a case with non-constant coefficient.

Example 4.1

Let P denote the operator $a^{-1} \frac{d}{dy} a \frac{d}{dy}$, where the function a is given by

$$a(y) = \begin{cases} a_+ & \text{for} \quad y \geq 0 \\ a_- & \text{for} \quad y < 0 \end{cases}$$

with a_+ and a_- being two positive constants. (This is the operator arising in the numerical examples of [13].)

It is not difficult to see that the following set of functions is a basis for $N(P^N)$:

$$\phi_0 = 1 \ , \qquad \phi_1 = \frac{1}{a(y)} y$$

$$\phi_{2k} = \int_{-1}^{y} \ell_{2k-1}(t)dt \ , \qquad \phi_{2k+1} = \frac{1}{a(y)} \int_{-1}^{y} \ell_{2k}(t)dt \qquad 1 \leq k \leq N-1 \quad .$$

Here ℓ_k denotes the Legendre polynomial or order k. Performing the Gram-Schmidt orthogonalization on the set $\phi_0, \phi_1, \ldots, \phi_{2N-2}, \phi_{2N-1}$ (in that sequence), in the inner-product $\langle u,v\rangle_a = \int_{-1}^{1} u(y)v(y)a(y)dy$, we end up with a new set of functions $\psi_0, \psi_1, \ldots, \psi_{2N-2}, \psi_{2N-1}$. ψ_k is a piecewise polynomial of degree k. Let L_a denote the operator $-a^{-1}(y) \frac{d}{dy} a(y)(1-y^2) \frac{d}{dy}$, it is then clear that

$$L_a \psi_{2k} = \sum_{j=0}^{k} \lambda_{j,k} \ \psi_{2j} \quad 0 \leq k \leq N-1 \quad ,$$

$$\mathcal{L}_a \psi_{2k-1} = \sum_{j=1}^{k} \tilde{\lambda}_{j,k}\, \psi_{2j-1} \qquad 1 \le k \le N \quad .$$

Now we have because of the orthogonality of the ψ_k's and the fact that $\mathcal{L}_a$ is self-adjoint in $L^2([-1,1],a(y)dy)$

$$\langle \mathcal{L}_a \psi_k, \psi_j \rangle_a = \langle \psi_k, \mathcal{L}_a \psi_j \rangle_a = 0 \qquad \text{for} \quad j < k$$

i.e., $$\mathcal{L}_a \psi_k = \lambda_k \psi_k \qquad \text{for any} \quad k \; .$$

It immediately follows that $\lambda_k = k(k+1)$.

It also follows, since $\{\psi_k\}_{k=0}^{\infty}$ is dense in $L^2([-1,1])$, that $\{\lambda_k, \psi_k\}_{k=0}^{\infty}$ is a complete set of eigenvalues and eigenfunctions for $\mathcal{L}_a$.

As in the proof of Theorem 4.3 we now get that

$$\inf_{v \in \mathcal{N}(P^N)} ||u-v||_{L^2} \le CN^{-2t}$$

$$\Updownarrow$$

$$u \in (\mathcal{D}(\mathcal{L}_a^p);\ \mathcal{D}(\mathcal{L}_a^q))_{s,\infty}$$

for any $0 \le p < t < q$, and $0 < s < 1$ chosen such that $t = p(1-s) + qs$. In summary, we have found a singular operator $\mathcal{L}_a$ that characterizes the rate of approximation with functions in $\mathcal{N}(P^N)$ the same way that the Legendre operator does with polynomials.

5. The Constant Coefficient Case - Dimensions Higher Than 1

In this section we prove a result relating to the dimensionally reduced solutions introduced in [13]. We give a characterization of the regularity needed for a certain rate of approximation. The approximating functions are of the form $\sum_{j=0}^{N} w_j(\underline{x})p_j(y)$ where $w_j \in H^1(\omega)$ and p_j is a polynomial of degree j, i.e., the operator P has constant coefficients. ω as before denotes a domain in $\mathbb{R}^n$ with a Lipschitz boundary and y ranges over $[-1,1]$.

In the proof of the main result in this section the following lemma will be very useful.

Lemma 5.1

Let $H_1 \subseteq H_0$ be two Banach spaces with norms $||\cdot||_1$ and $||\cdot||_0$ respectively.

Let $\{V_N\}_{N=0}^{\infty}$ be an increasing sequence of subspaces of H_1 and let β be a positive number. We assume that the following implication holds

$$u \in H_0 \quad \text{and} \quad \inf_{q \in V_N} ||u-q||_0 \leq C_u(N+1)^{-\beta}, \quad \forall N \geq 0$$

$$\Downarrow$$

$$u \in H_1 \quad \text{and} \quad ||u||_1 \leq C(C_u + ||u||_0)$$

for some C independent of u. (C_u here is not generic, it is the same constant in the two inequalities.)

As a result of this it follows that for any $0 < \theta < 1$

$$u \in H_0 \quad \text{and} \quad \inf_{q \in V_N} ||u-q||_0 \leq C_u(N+1)^{-\theta\beta}, \quad \forall N \geq 0$$

$$\Downarrow$$

$$u \in (H_0;H_1)_{\theta,\infty} \quad \text{and} \quad ||u||_{\theta,\infty} \leq C(C_u + ||u||_0)$$

for some C independent of u. (As before C_u is not generic.)

Proof

Let $0 < \theta < 1$ and assume that there exists a sequence of elements $q_N \in V_N$, $N \geq 0$, such that

$$||u-q_N||_0 \leq C_u(N+1)^{-\theta\beta} \quad .$$

Define

$$r_0 = q_1$$

$$r_m = q_{2^m} - q_{2^{m-1}} \qquad m \geq 1 \quad ,$$

then $$||u - \sum_{m=0}^{k} r_m||_0 = ||u-q_{2^k}||_0 \leq C_u\, 2^{-k\theta\beta} \;.$$

At the same time

$$||r_0||_0 \leq C_u + ||u||_0 \qquad \text{and}$$

$$||r_m||_0 \leq ||u-q_{2^m}||_0 + ||u-q_{2^{m-1}}|| \leq C_{\theta,\beta}C_u\, 2^{-m\theta\beta} \;, \qquad m \geq 1 \quad .$$

That is, $r_m \in V_{2^m}$ $m \geq 1$, and

$$||2^{m\beta(\theta-1)}r_m||_0 \leq C_{\theta,\beta}(C_u + ||u||_0)2^{-m\beta} \quad .$$

From the first implication in the statement of this theorem it follows that

$$||2^{m\beta(\theta-1)}r_m||_1 \leq C_{\theta,\beta}\,(C_u + ||u||_0 + ||2^{m\beta(\theta-1)}r_m||_0)$$

or $$||r_m||_1 \leq C_{\theta,\beta}(C_u + ||u||_0)\cdot 2^{m\beta(1-\theta)} .$$

We therefore get

$$||\sum_{m=0}^{k} r_m||_1 \leq \sum_{m=0}^{k} ||r_m||_1 \leq C_{\theta,\beta}(C_u + ||u||_0)\cdot 2^{k\beta(1-\theta)} .$$

If we define $s_k = 2^{-k\beta}$ the following inequality now holds

$$s_k^{-\theta}(||u - \sum_{m=0}^{k} r_m||_0 + s_k||\sum_{m=0}^{k} r_m||_1) \leq C_{\theta,\beta}(C_u + ||u||_0) .$$

Since $s_k \to 0$ for $k \to \infty$ this proves that

$$u \in (H_0;H_1)_{\theta,\infty} \quad \text{with}$$

$$||u||_{\theta,\infty} \leq C_{\theta,\beta}(C_u + ||u||_0) .$$

$\square$

Let us introduce the spaces

$$K^R = \{u \in \mathcal{D}(\tilde{L}^R) \mid \frac{\partial}{\partial x_i} u \in \mathcal{D}(\tilde{L}^R) \qquad i = 1,\ldots,n\} ,$$

$\tilde{L}$ here denotes $-\frac{\partial}{\partial y}(1-y^2)\frac{\partial}{\partial y}$ considered as an operator $L^2(\omega \times [-1,1]) \supseteq \mathcal{D}(\tilde{L}) \to L^2(\omega \times [-1,1])$, and R is a non-negative integer.

V_N denotes the space $\{\sum_{j=0}^{N} w_j(\underline{x})p_j(y) \mid w_j \in H^1(\omega)\}$ where p_j is a polynomial of degree j , $j \geq 0$. We are then able to give the following characterization of approximation by the spaces V_N in the H^1-norm.

Theorem 5.1

Let α be a given positive number. If $u \in (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty}$ for some integer $R > \alpha$ and $R > 2\alpha/\varepsilon$, where ε is a positive number, then there exists a constant C such that

$$\inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C(N+1)^{-2\alpha+\varepsilon}, \quad \forall N \geq 0 \quad .$$

On the other hand if for some $\varepsilon > 0$ there exist a constant C such that

$$\inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C(N+1)^{-2\alpha-\varepsilon}, \quad \forall N \geq 0 ,$$

then

$$u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty}$$

Before we proceed with the proof of Theorem 5.1, let us state a corollary that immediately follows from this theorem.

Modulo ε this is the equivalent of Theorem 4.3 in more than one dimension.

Corollary 5.1

Let α be a given positive integer. If $u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty}$, then for any $\varepsilon > 0$ there exist a constant C_ε such that

$$\inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C_\varepsilon (N+1)^{-2\alpha+\varepsilon} , \quad N \geq 0 .$$

On the other hand if for some $\varepsilon > 0$ there exists a constant C such that

$$\inf_{q\in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \le C(N+1)^{-2\alpha-\varepsilon}, \quad \forall N \ge 0 ,$$

then

$$u \in \bigcap_{R\in\mathbb{N},R>\alpha} (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty} .$$

Proof of Theorem 5.1

Assume that $u \in K^R$ and R is an integer > 1. Let v_N denote the orthogonal projection of $\tilde{L}u$ onto $\{\sum_{j=0}^{N} w_j(\underline{x})p_j(y) | w_j \in L^2(\omega)\}$ in the $L^2(\omega \times [-1,1])$ inner product. Then it is clear that $\frac{\partial}{\partial x_i} v_N$ is the L^2 projection of $\tilde{L}\frac{\partial}{\partial x_i} u$ onto the same subspace. From Theorem 4.3 we immediately get

$$\sum_{i=1}^{n} ||\tilde{L}\frac{\partial}{\partial x_i}(u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||\tilde{L}(u-v_N^*)||_{L^2(\omega \times [-1,1])} \le$$

$$\le C_R N^{2-2R}\left(\sum_{i=1}^{n} ||\frac{\partial}{\partial x_i} u||_{\mathcal{D}(\tilde{L}^R)} + ||u||_{\mathcal{D}(\tilde{L}^R)}\right)$$

for any function $v_N^* \in V_N$ with $\tilde{L}v_N^* = v_N$. Now choosing v_N^* so that also $\int_{-1}^{1} v_N^*(\underline{x},y)dy = \int_{-1}^{1} u(\underline{x},y)dy$ for any $\underline{x} \in w$ (this is obviously possible) it follows that

$$\sum_{i=1}^{n}||\frac{\partial}{\partial x_i}(u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||u-v_N^*||_{L^2(\omega \times [-1,1])} + ||\frac{\partial}{\partial y}(u-v_N^*)||_{L^2(\omega \times [-1,1])} \le$$

$$\le C\left(\sum_{i=1}^{n} ||\tilde{L}\frac{\partial}{\partial x_i}(u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||\tilde{L}(u-v_N^*)||_{L^2(\omega \times [-1,1])}\right) \le$$

$$C_R N^{2-2R}\left(\sum_{i=1}^{n} ||\frac{\partial}{\partial x_i} u||_{\mathcal{D}(\tilde{L}^R)} + ||u||_{\mathcal{D}(\tilde{L}^R)}\right) = C_R N^{2-2R}||u||_{K^R}$$

Using interpolation on this result we get that

$$\inf_{q\in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C_R N^{-2\alpha+\frac{2\alpha}{R}} |||u|||_{\alpha/R,\infty} ,$$

where $|||u|||_{\alpha/R,\infty}$ denotes the norm on $(H^1(\omega \times [-1,1];K^R)_{\alpha/R,\infty}$. Since $2\alpha/\varepsilon < R$, i.e. $2\alpha/R < \varepsilon$, *the direct part of this theorem immediately follows.*

Let us now give a proof of the second part of the theorem. If $\inf_{q\in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq CN^{-2R-\frac{R\varepsilon}{\alpha}}$ for some $R > \alpha$, then as in the proof of Theorem 4.3 it easily follows that

$$u \in \mathcal{D}(\tilde{L}^R) \quad \text{and} \quad \frac{\partial}{\partial x_i} u \in \mathcal{D}(\tilde{L}^R) \quad ,$$

i.e. $u \in K^R$. If we apply Lemma 5.1 with $H_1 = K^R$, $H_0 = H^1(\omega \times [-1,1])$ and $\theta = \alpha/R$, we then get that

$$\inf_{q\in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C(N+1)^{-2\alpha-\varepsilon}$$

implies

$$u \in (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty} ,$$

for any integer $R > \alpha$, i.e.

$$u \in \bigcap_{R\in\mathbb{N},R>\alpha} (H^1(\omega \times [-1,1]);K^R)_{\alpha/R,\infty}$$

$\square$

For the conclusion of this section let us give a simple example that shows the practical usefulness of Theorem 5.1 (or corollary 5.1).

Example 5.1

Let ω be the interval $[0,1]$. Let γ be a positive number and let (r,θ) denote polar coordinates around the point $(1,1)$. We then consider functions of the type $u = r^{\gamma}\phi(\theta)$, where ϕ is an element of $C^{\infty}([0,\pi/2])$.

It is not difficult to prove that

$$u \in \bigcap_{R \in \mathbb{N}, R > \tau} (H^1(\omega \times [-1,1]); K^R)_{\tau/R,\infty}$$

for $0 < \tau \leq \gamma$, and that for a general choice of ϕ this is not so for any $\tau > \gamma$ (if $\gamma \notin \mathbb{N}$, then this is not so for any ϕ and $\tau > \gamma$ except $\phi = 0$). By an application of corollary 5.1 we therefore get that

$$\inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C_\varepsilon (N+1)^{-2\gamma+\varepsilon}, \quad \forall N \geq 0 \quad ,$$

for any $\varepsilon > 0$, and at the same time that for a general choice of ϕ (or for any $\phi \neq 0$ in the case $\gamma \notin \mathbb{N}$) there exist no $\varepsilon > 0$ and C_ε such that

$$\inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C_\varepsilon (N+1)^{-2\gamma-\varepsilon}, \quad \forall N \geq 0 \quad .$$

A function of the type $r^{\gamma}\phi(\theta)$ is a typical example of a corner-singularity as arising from the solution of an elliptic boundary value problem.

Theorem 5.1 (or corollary 5.1) is thus well suited to predict the optimal order of convergence (modulo ε) that one can in general expect by dimensional reduction of elliptic boundary value problems.

A result like this could not have been obtained by using the a-priori knowledge of the regularity of solutions to elliptic boundary value problems in terms of standard Sobolev spaces.

6. A Simple Example of Dimensional Reduction

Let us consider the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad]0,1[\times]-1,1[$$

$$u = 0 \quad \text{for} \quad x = 0,1$$

$$\frac{\partial u}{\partial n} = g(x) \quad \text{for} \quad y = \pm 1$$

(n is the outward normal).

From [13] it follows that the optimal choice of basis functions for dimensional reduction in this case is the polynomials. $\overset{\circ}{V}_N$ denotes the set $\{ \sum_{j=0}^{N} w_j(x)\, p_j(y) \mid w_j \in \overset{\circ}{H}{}^1([0,1]) \}$, where p_j is a polynomial of degree j.

Let u_N denote the projection of u onto $\overset{\circ}{V}_N$ in the inner product $B(\phi,\psi) = \int_0^1 \int_{-1}^1 \left(\frac{\partial\phi}{\partial y} \frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial x} \frac{\partial\psi}{\partial x} \right) dy dx$. It is clear that

$$\inf_{q \in \overset{\circ}{V}_N} \| u - q \|^2_{H^1([0,1]\times[-1,1])} \le C\, B(u-u_N, u-u_N) \le$$

$$\le C \inf_{q \in \overset{\circ}{V}_N} \| u - q \|^2_{H^1([0,1]\times[-1,1])} ,$$

and hence that the energy error $B(u-u_N,\, u-u_N)$ is asymptotically in N equivalent to the square of the distance (in H^1) from u to $\overset{\circ}{V}_N$.

If g has the Fourier series

$$g(x) = \sum_{k=1}^{\infty} g_k \sin k\pi x ,$$

then it immediately follows that u is given by

$$u(x,y) = \sum_{k=1}^{\infty} \frac{\cosh(k\pi y)}{\sinh(k\pi)} \frac{g_k}{k\pi} \sin k\pi x \quad .$$

In terms of regularity of u it is not difficult to prove that this formula leads to the following three results:

i) $\forall\alpha \geq 0\colon \quad \sum_{k=1}^{\infty} g_k^2\, k^{2\alpha} < \infty \Longleftrightarrow u \in H^{3/2+\alpha}([0,1]x[-1,1])$

ii) $\forall\alpha \geq 0\colon \quad \sum_{k=1}^{\infty} g_k^2\, k^{2\alpha} < \infty \Longrightarrow \begin{cases} u \in \bigcap_{R\in\mathbb{N},\, R>\theta} (H^1([0,1]x[-1,1]);K^R)_{\theta/R,\infty} \\ \text{with } \theta = \alpha + 1/2 \end{cases}$

iii) $\forall\alpha \geq 0,\ \varepsilon > 0\colon \quad \left. \begin{array}{r} u \in \bigcap_{R\in\mathbb{N},\, R>\theta} (H^1([0,1]x[-1,1]);K^R)_{\theta/R,\infty} \\ \text{with } \theta = \alpha + 1/2 + \varepsilon . \end{array} \right\} \Longrightarrow \sum_{k=1}^{\infty} g_k^2\, k^{2\alpha} < \infty .$

We consider two different choices for g

$$g(x) = \pi/4$$
$$g(x) = x(x-1)$$

For the first choice of g it follows that

$$\sum_{k=1}^{\infty} g_k^2\, k^{2\theta} < \infty \quad \text{for any} \quad \theta < 1/2 \text{ , and}$$

$$\sum_{k=1}^{\infty} g_k^2\, k = \infty \quad ,$$

similarly for the second choice

$$\sum_{k=1}^{\infty} g_k^2 k^{2\theta} < \infty \text{ for any } \theta < 5/2 \text{ , and}$$

$$\sum_{k=1}^{\infty} g_k^2 k^5 = \infty \quad .$$

Corollary 5.1 together with the regularity results ii) and iii) now ensure that

In the case $g(x) = \pi/4$

$B(u-u_N, u-u_N)$ will converge to zero faster than $N^{-4+\varepsilon}, \forall \varepsilon > 0$, but on the other hand slower than $N^{-4-\varepsilon}, \forall \varepsilon > 0$.

In the case $g(x) = x(x-1)$

$B(u-u_N, u-u_N)$ will converge to zero faster than $N^{-12+\varepsilon}, \forall \varepsilon > 0$, but on the other hand slower than $N^{-12-\varepsilon}, \forall \varepsilon > 0$.

Figures 1 and 2 show the actual computed values of $B(u-u_N, u-u_N)$ as a function of N in the two different cases. Note that the asymptotic rate of convergence is obtained already for a fairly small number of polynomials.

For details concerning the computation of the u_N's see [13].

Instead of using Corollary 5.1 to obtain information about the rate of convergence we could have used the regularity result i) and a 2-dimensional version of Theorem 4.1. This way we could at most have predicted convergence of the order of $N^{-2+\varepsilon}$ and $N^{-6+\varepsilon}$ respectively, i.e. only half the actual convergence rate.

In [13] we considered the same boundary value problem as here, only it was on the domain $[0,1]x[-h,h]$ for some $h > 0$, and not on $[0,1]x[-1,1]$. From

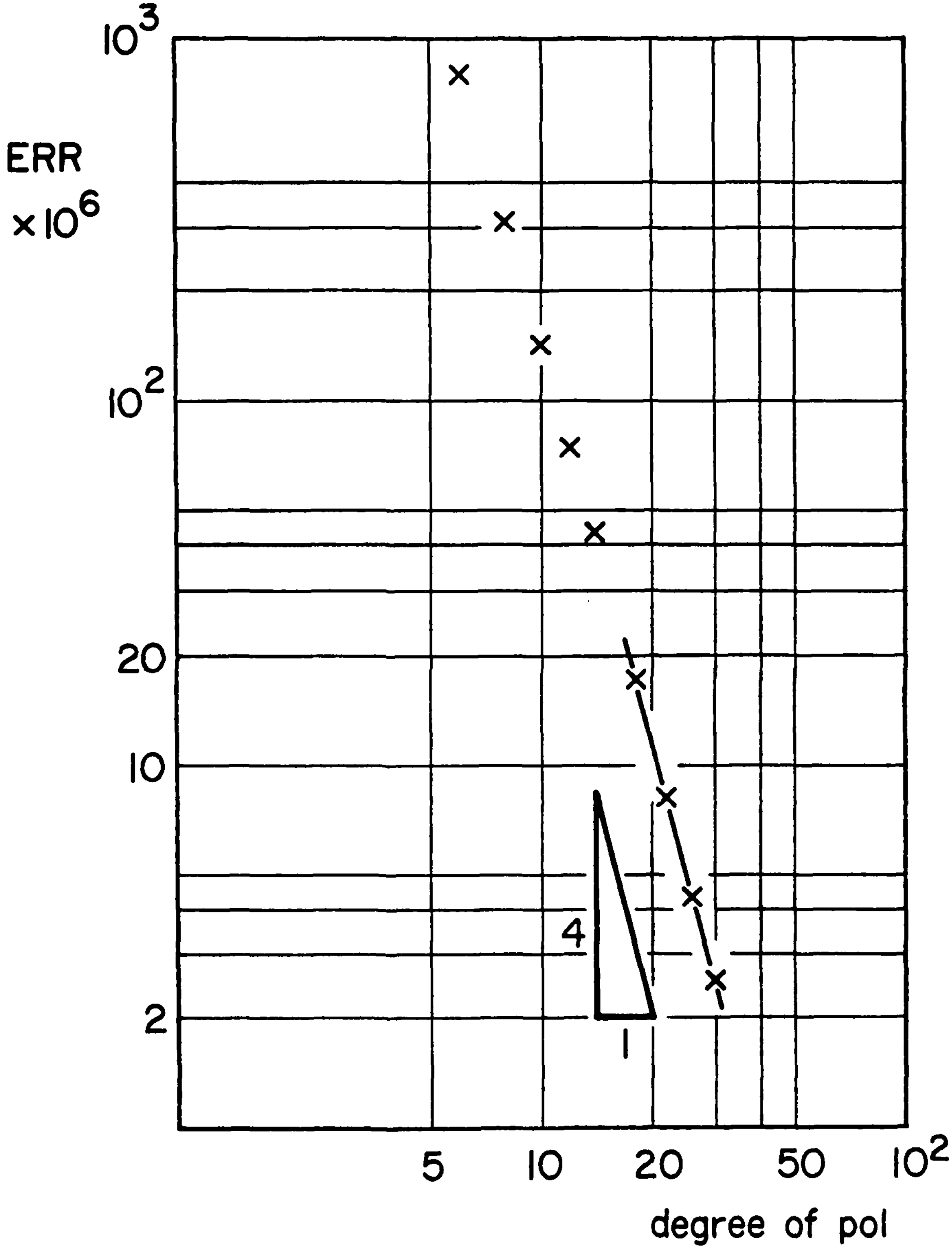

Fig. 1 Energy error $x\ 10^6$ with $g(x) = \pi/4$, $h = 1$.

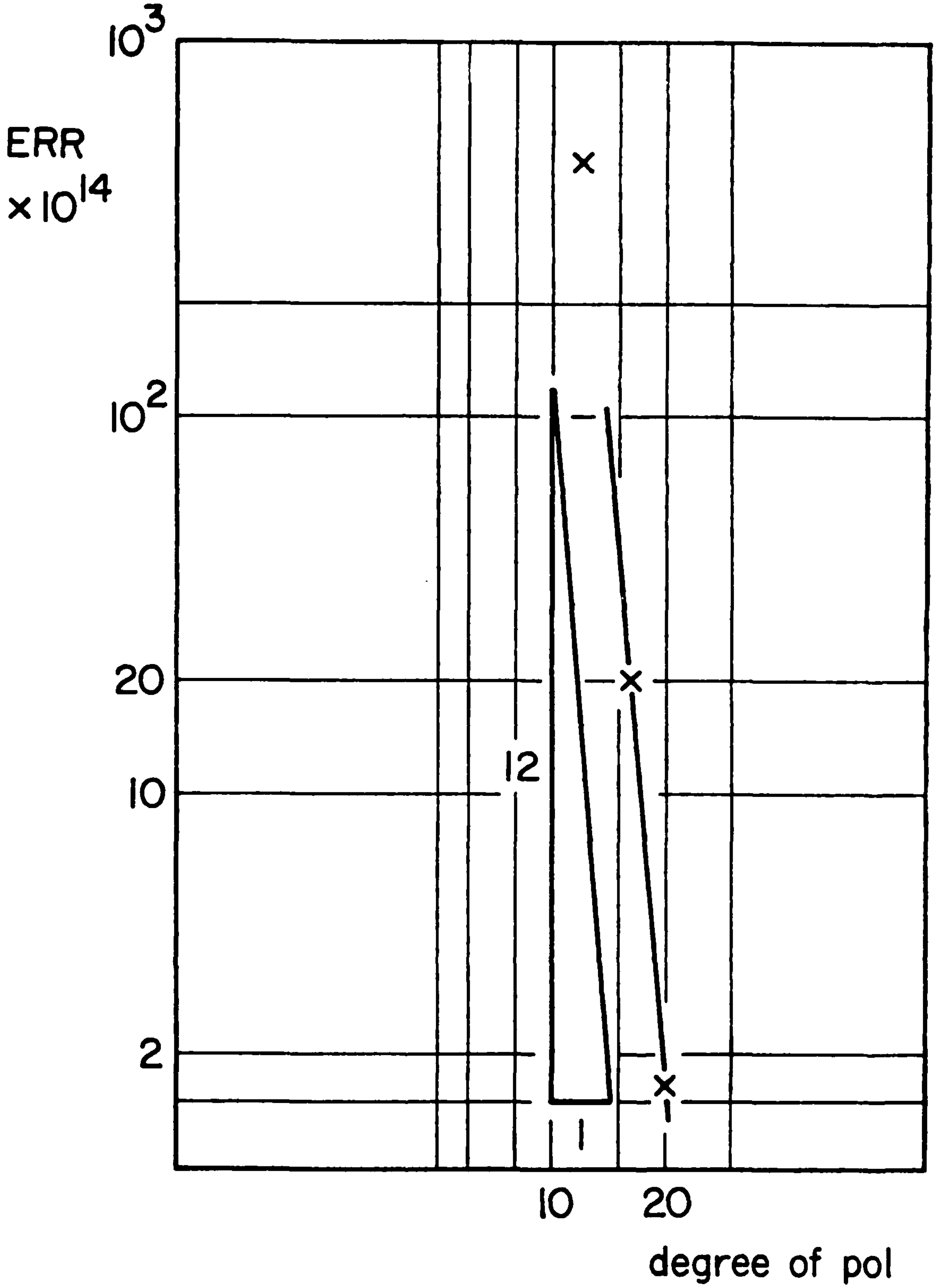

Fig. 2 Energy error $\times 10^{14}$ with $g(x) = x(x-1)$, $h = 1$.

the computational results there it follows that for a fixed $N \geq 2$, $B(u-u_N^h, u-u_N^h)$ behaves like h^2, $h \to 0$, in the case where $g(x) = \pi/4$. (u_N^h is the projection of u onto $\{ \sum_{j=0}^{N} w_j(x)\, p_j(y/h) \mid w_j \in \overset{0}{H}{}^1([o,1]) \}$). Comparing this to the result obtained here for $g(x) = \pi/4$ it is seen that using N polynomials, $y \in [-1,1]$, is in some sense equivalent to having a domain of thickness $1/N^2$. A similar feature has been noticed by comparison of the standard h-version of the F.E.M. with the so-called p-version (cf. [2]).

In this example we used slight variations of the approximation results proved in sections 4 and 5, namely with fixed boundary conditions $\equiv 0$ at $x = 0,1$. The proofs of these results follow immediately from the proofs of the similar results with no boundary conditions.

Appendix

In sections 2 and 3, we used some results concerning the eigenvalues and eigenfunctions of the boundary value problem

$$-\frac{d}{dy}\, a \frac{d}{dy}\, u = \lambda u$$

$$u(-1) = u(1) = 0 \quad .$$

a here is a function in $L^{\infty}([-1,1])$ such that $\exists$ a constant a_o with $0 < a_o \leq a(y)$. From the theory of Sturm-Liouville systems it immediately follows that the eigenvalues (repeated according to multiplicity) form a sequence:

$$0 < \lambda_o \leq \lambda_1 \leq \; . \; . \; \leq \lambda_m \leq \lambda_{m+1} \; . \; .$$

with $+\infty$ as the only limit point.

Lemma A.1

With notation as above

$$\lambda_m \neq \lambda_{m'} \quad \text{for} \quad m \neq m' \quad ,$$

i.e. the eigenvalues are all simple.

Proof

Assume that for some $m \neq m'$ $\lambda_m = \lambda_{m'}$. This means that the eigenvalue $\lambda = \lambda_m (= \lambda_{m'})$ has multiplicity > 1 . Let u and $\tilde{u}$ be two linearly independent eigenvectors corresponding to λ , and let $v = cu + d\tilde{u}$ be a nontrivial linear combination with the property that $a\frac{dv}{dy} = 0$ for $y = -1$ (such one obviously

exists). The function v is then a solution to the initial-value problem

$$-\frac{d}{dy} a \frac{d}{dy} v = \lambda v \qquad \text{in } [-1,1]$$

$$v = a\frac{d}{dy} v = 0 \qquad \text{for } y = -1 \quad ,$$

and because of the uniqueness of solutions to this problem, it follows that $v = 0$. Since v is a nontrivial linear combination of u and $\tilde{u}$ this shows that u and $\tilde{u}$ are linearly dependent. We therefore have arrived at a contradiction, i.e., $\lambda_m \neq \lambda_{m'}$ for $m \neq m'$.

$\square$

It is well known that there exist constants $0 < C_1$ and $0 < C_2$ such that $C_1(m+1)^2 \leq \lambda_m \leq C_2(m+1)^2$.

By imposing an extra smoothness requirement on a we can obtain a much more detailed statement.

<u>Lemma A.2</u>

If $a \in C^2([-1,1])$, then

$$\lambda_m = (\pi/\ell)^2 \cdot (m+1)^2 + O(1) \quad ,$$

$$\text{where } \ell = \int_{-1}^{1} (a(y))^{-1/2}\, dy \quad .$$

A proof of Lemma A.2 is found in chapter 4 of [11], and shall not be repeated here.

Let $\{u_m\}_{m=0}^{\infty}$ denote a sequence of normalized eigenfunctions, u_m corresponding to λ_m . Let f_0 be given as in section 2, namely

$$f_0(y) = \int_{-1}^{y} \frac{1}{a(s)}\, ds \quad .$$

Lemma A.3

The function f_0 has the expansion

$$f_0 = \sum_{m=0}^{\infty} \alpha_m u_m \quad ,$$

where $\alpha_m \neq 0$ for every m .

Proof

That f_0 has a unique expansion is wellknown. The coefficient α_m is given by

$$\alpha_m = \int_{-1}^{1} \int_{-1}^{y} \frac{1}{a(s)} ds \quad u_m(y) \, dy$$

Now assume that for some value of $m = m_o$ $\alpha_{m_o} = 0$, i.e.

$$\int_{-1}^{1} \int_{-1}^{y} \frac{1}{a(s)} \, ds \quad u_{m_o}(y) dy = 0 \quad .$$

Since $u_{m_o}(y) = -\frac{1}{\lambda_{m_o}} \frac{d}{dy} a \frac{d}{dy} u_{m_o}$ we get that

$$\int_{-1}^{1} \int_{-1}^{y} \frac{1}{a(s)} ds \left[\frac{d}{dy} a \frac{d}{dy} u_{m_o} \right] (y) \quad dy = 0 \quad .$$

Performing an integration by parts this yields

$$\int_{-1}^{1} \frac{1}{a(s)} \, ds \left[a \frac{d}{dy} u_{m_o} \right] (1) - \int_{-1}^{1} \frac{d}{dy} u_{m_o}(y) \quad dy = 0 \quad ,$$

and the last integral here vanishes due to the fact that $u_{m_o}(1) = u_{m_o}(-1) = 0$. We therefore conclude that

$$a \frac{d}{dy} u_{m_o} = u_{m_o} = 0 \qquad \text{for} \quad y = 1 \quad .$$

On the other hand u_{m_o} satisfies the differential equation

$$\frac{d}{dy} a \frac{d}{dy} u_{m_o} + \lambda_{m_o} u_{m_o} = 0 \qquad \text{in} \quad [-1,1] \quad .$$

Because of uniqueness of solutions to the initial-value problem, this implies that $u_{m_o} = 0$. We have thus arrived at a contradiction, meaning that $\alpha_m \neq 0$ for every m.

$\square$

Again, by imposing an extra smoothness requirement on a, we obtain a very detailed result concerning the decay-properties of the α_m's

Lemma A.4

If $a \in C^2([-1,1])$, then $\exists$ constants $0 < C_1$ and $0 < C_2$ such that

$$\frac{C_1}{m+1} \leq | \alpha_m | \leq \frac{C_2}{m+1} \quad \text{for all} \quad m \quad .$$

Proof

From [11] p. 176 we get the following asymptotic formula for $u_m(y)$

$$u_m(y) = D_m(a(y))^{-1/4} \left[\sin\left(\frac{(m+1)\pi}{\ell} \xi\right) - \right.$$

$$\left. - \frac{1}{m+1} T(\xi) \cos\left(\frac{(m+1)\pi}{\ell} \xi\right)\right] + O((m+1)^{-2}) \quad ,$$

where $\xi(y) = \int_{-1}^{y} (a(s))^{-1/2} ds$ and $\ell = \xi(1) = \int_{-1}^{1} (a(s))^{-1/2} ds$. The function T is in C^1 and the constants D_m satisfy

$$\exists\ D \text{ (independent of } m \text{) such that}$$

$$1/D \leq |D_m| \leq D \qquad \text{for all } m .$$

Also, $O(\cdot)$ here means uniformly in y . Let us now calculate α_m :

$$\alpha_m = \int_{-1}^{1} f_0(y)\ u_m(y)dy = \int_{-1}^{1}\int_{-1}^{y} \frac{1}{a(s)} ds\ \ u_m(y)dy =$$

$$I_1 + I_2 + O((m+1)^{-2}) \qquad .$$

I_2 denotes the integral

$$-\frac{1}{m+1} D_m \int_{-1}^{1} \left(\int_{-1}^{y} \frac{1}{a(s)} ds \right) (a(y))^{-1/4}\ T(\xi)\cos\left(\frac{(m+1)\pi}{\ell}\ \xi \right) dy \ .$$

By a change of variables from y to ξ and an integration by parts, it immediately follows that I_2 is $O((m+1)^{-2})$.

I_1 is given as

$$D_m \int_{-1}^{1} \left(\int_{-1}^{y} \frac{1}{a(s)}\ ds \right) (a(y))^{-1/4}\ \sin\left(\frac{(m+1)\pi}{\ell}\ \xi \right) dy \quad .$$

By a change of variables from y to ξ and an integration by parts we get that

$$I_1 = D_m \left(\int_{-1}^{1} \frac{1}{a(s)} ds \right) \ell \; (a(1))^{1/4} \cdot \frac{1}{(m+1)\pi} \cdot (-1)^{m+1} + O((m+1)^{-2}) \quad .$$

This immediately implies the existence of two constants $0 < C_1$ and $0 < C_2$ such that

$$\frac{C_1}{m+1} \leq |\alpha_m| \leq \frac{C_2}{m+1}$$

for m sufficiently large. Now combining with Lemma A.3 and possibly changing the constants C_1 and C_2 we get the desired result.

<u>References</u>

[1] I. Babuška, Error-bounds for finite element method. Numer. Math. <u>16</u>, pp.322-33.

[2] I. Babuška, B. A. Szabo, I. N. Katz, The p-version of the finite element method. Report WU/CCM-79/1. Washington University in St. Louis, May 1979; (to appear in SIAM Journal of Num. Anal.).

[3] M. S. Baouendi, C. Goulaouic, Régularité et theorie spectrale pour une classe d'opérateurs elliptiques dégénérés. Arch. Rational Mech. Anal. <u>34</u> (1969), pp.361-79.

[4] J. Bergh, J. Löfström, Interpolation spaces. Springer, 1976.

[5] P. Destuynder, Doctoral Thesis. Paris VI, January 1980.

[6] V. K. Dzjadyk, An introduction to the theory of uniform approximation of functions by polynomials. Nauka, 1977 (in Russian).

[7] D. G. Gordeziani, Accuracy of a variant in the theory of thin shells. Sov. Phys. Dokl. <u>19</u> (1974), pp.385-86.

[8] J. L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I. Springer, 1972.

[9] G. G. Lorentz, Approximation of functions. Holt, Rinehart and Winston, 1966.

[10] S. M. Nikol'skiĭ, Approximation of functions of several variables and imbedding theorems. Springer, 1975.

[11] F. G. Tricomi, Differential equations. Blackie & Son, 1961.

[12] M. Vogelius, Ph.D. Thesis. University of Maryland, December 1979.

[13] M. Vogelius, I. Babuška, On a dimensional reduction method. The optimal selection of basis functions. Tech. Note BN-939, University of Maryland, April 1980.

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